Analytical invariants of quasi-ordinary hypersurface singularities associated to divisorial valuations

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Abstract. We study an analytically irreducible algebroid germ $(X,0)$ of complex singularity by considering the filtrations of its analytic algebra, and their associated graded rings, induced by the *divisorial valuations* associated to the irreducible components of the exceptional divisor of the normalized blow-up of the normalization $(\bar{X},0)$ of $(X,0)$, centered at the point $0 \in \bar{X}$. If $(X,0)$ is a quasi-ordinary hypersurface singularity, we obtain that the associated graded ring is a $\mathbf{C}$-algebra of finite type, namely the coordinate ring of a non necessarily normal affine toric variety of the form $Z^\Gamma = \mathrm{Spec}\mathbf{C}[\Gamma]$, and we show that the semigroup Γ is an analytical invariant of $(X,0)$. This provides another proof of the analytical invariance of the *normalized characteristic monomials* of $(X,0)$ (see [8], [19] and [24]). If $(X,0)$ is the algebroid germ of non necessarily normal toric variety, we apply the same method to prove a local version of the isomorphism problem for algebroid germs of non necessarily normal toric varieties (see [15], for the algebraic case).

Introduction

We study a class of algebroid hypersurface singularities called *quasi-ordinary*. These singularities arise classically in Jung's approach to analyze surface singularities by using a finite projection to a smooth surface (see [16], [27]). Quasi-ordinary hypersurface singularities are parametrized by *quasi-ordinary branches*, certain class of fractional power series in several variables having a finite set of *distinguished* or *characteristic monomials*, generalizing the *characteristic pairs* associated to a plane branch (see [28]). These monomials determine many features of the geometry and topology of the singularity, for instance in the analytically irreducible case they define a complete invariant of the embedded topological type (see Lipman's and Gau's works [19] and [8]). This characterization implies the analytical invariance of the characteristic monomials suitably *normalized* by an inversion lemma of Lipman (see [8], Appendix and [11]). In the case of algebroid surfaces the analytical invariance of the normalized characteristic monomials was proved by Lipman and also by Luengo, by building canonical sequences of monoidal and quadratic transformations which desingularize the germ, however these methods do not admit generalizations to quasi-ordinary hypersurfaces of dimension ≥ 3 (see [18] and [20]).

In what follows $(S,0)$ denotes the algebroid germ of an analytically irreducible quasi-ordinary hypersurface. In [12], the first author introduced a semigroup Γ associated to a fixed quasi-ordinary branch

parametrizing the germ $(S,0)$ and showed, using the characterization of the topological type, that different quasi-ordinary branches parametrizing $(S,0)$ provide isomorphic semigroups. In this Note we prove that the semigroup Γ is an analytical invariant of the algebroid germ $(S,0)$, without passing by Lipman's and Gau's topological approach. The isomorphism class of the semigroup Γ determines the normalized characteristic monomials of the germ $(S,0)$ (see [12], [11] or [23]). Our result provides a proof of the analytical invariance of the normalized characteristic monomials of the quasi-ordinary hypersurface $(S,0)$.

Popescu-Pampu has given another proof of the analytical invariance of the normalized characteristic monomials of the quasi-ordinary hypersurface $(S,0)$ (see [24] and [23]). In his work [24], which has a flavor similar to that of the surface case [23], the structure of the singular locus of $(S,0)$ is used to build a sequence of blow-ups of $(\mathbf{C}^d,0)$ (where $d=\dim S$), in terms of the presentation of the normalization of $(S,0)$ as a quotient singularity. This sequence is used to define a semigroup Γ' depending only on the germ $(S,0)$. He proves, using a fixed quasi-ordinary branch parametrizing $(S,0)$, that the semigroup Γ' is a linear projection of the semigroup Γ, eventually, of rank less than the rank of Γ. The normalized characteristic monomials are recovered from the semigroup Γ' and from analysis of the equisingularity class of certain plane sections of $(S,0)$, this latter technique is also essential in Gau's work [8].

Instead of defining an intrinsic semigroup associated to the singularity, we study certain filtrations of its analytic algebra, and their associated graded rings. This approach is inspired by the description, due to Lejeune-Jalabert, of the semigroup of a plane branch in terms of the graded ring associated to the filtration of its local ring by the powers of the maximal ideal of its integral closure (see [26] page 161). If $(X,0)$ is an analytically irreducible algebroid germ so is its normalization $(\bar{X},0)$. We can study the singularity $(X,0)$ by considering the filtrations, and their associated graded rings, induced by the *divisorial valuations* associated to the irreducible components of the exceptional divisor of the normalized blow-up of $(\bar{X},0)$ centered at the point $0\in\bar{X}$. In general graded rings associated to divisorial valuations on normal or regular local rings are not necessarily Noetherian (see examples by Cutkosky and Srinivas [5], page 557, and by Cossart, Galindo and Piltant in [4]). We apply this strategy successfully in two particular instances:

- If $(X,0)$ is the algebroid germ of a non necessarily normal affine toric variety of the form, $Z^{\Lambda}=\mathrm{Spec}\,\mathbf{C}[\Lambda]$ at its zero dimensional orbit 0, we obtain that the graded ring associated to $(X,0)$ by any of these divisorial valuations, is a $\mathbf{C}$-algebra of finite type, namely the coordinate ring of the affine toric variety Z^{Λ}. We apply a theorem of Gubeladze on the *isomorphism problem* for commutative monoid rings (see [15]) to recover the semigroup Λ from the toric variety Z^{Λ}. This means that the semigroup Λ is an analytic invariant of the algebroid germ $(X,0)$, and it does not depend on the choice of toric structure on $(X,0)$. A topological proof of this result, in the case of normal simplicial toric singularities, has been given by Popescu-Pampu in [25].

- If $(X,0)$ is the germ of quasi-ordinary hypersurface $(S,0)$ we obtain that the graded ring associated to $(S,0)$ by any of these divisorial valuations, is a $\mathbf{C}$-algebra of finite type, namely the coordinate ring of the affine toric variety $Z^{\Gamma}:=\mathrm{Spec}\,\mathbf{C}[\Gamma]$. The algebroid germ $(Z^{\Gamma},0)$ of the affine toric variety Z^{Γ} at its zero dimensional orbit 0, is of geometric significance to the hypersurface germ $(S,0)$, for instance both germs have the same normalization (see [13]), and any of these divisorial

valuations associates to $(S,0)$ and $(Z^\Gamma,0)$ a pair of isomorphic graded rings. The analytical invariance of the semigroup Γ follows simultaneously for the algebroid germs $(S,0)$ and $(Z^\Gamma,0)$. The same strategy provides a proof of the analytical invariance of the semigroup associated to a *toric quasi-ordinary hypersurface*. This kind of singularity belongs to certain class of branched coverings of a germ of normal affine toric variety at the special point, which are unramified over the torus, and plays an important role in the toric embedded resolutions of quasi-ordinary hypersurface singularities described in [13]. It is shown in [13] that the germs $(S,0)$ and $(Z^\Gamma,0)$ have simultaneous toric embedded resolutions, which are described from the properties of the semigroup Γ by a method inspired by that of Goldin and Teissier for plane branches (see [10]).

1 A reminder of toric geometry

We give some definitions and notations (see [22], [21], [6], [7] and [17] for proofs). If N is a lattice we denote by M the dual lattice, by $N_{\mathbf{R}}$ the real vector space spanned by N. A *rational convex polyhedral cone* σ in $N_{\mathbf{R}}$, a *cone* in what follows, is the set of non negative linear combinations of vectors $a^1,\dots,a^s \in N$. The cone σ is *strictly convex* if σ contains no linear subspace of dimension >0. We denote by $\overset{\circ}{\sigma}$ the *relative interior* of a cone σ. The *dual* cone $\sigma^\vee$ (resp. *orthogonal* cone $\sigma^\perp$) of σ is the set $\{w \in M_{\mathbf{R}} / \langle w,u\rangle \geq 0,$ (resp. $\langle w,u\rangle = 0$) $\forall u \in \sigma\}$. A *fan* Σ is a family of strictly convex cones in $N_{\mathbf{R}}$ such that any face of such a cone is in the family and the intersection of any two of them is a face of each. The *support* of the fan Σ is the set $\bigcup_{\sigma\in\Sigma}\sigma \subset N_{\mathbf{R}}$.

A non necessarily normal affine toric variety is of the form $Z^\Lambda = \operatorname{Spec}\mathbf{C}[\Lambda]$ where Λ is a subsemigroup of finite type of a lattice $\Lambda+(-\Lambda)$ which it generates as a group. The torus $Z^{\Lambda+(-\Lambda)}$ is an open dense subset of Z^Λ, which acts on Z^Λ, and this action extends the action of the torus on itself by multiplication. The semigroup Λ spans the cone $\mathbf{R}_{\geq 0}\Lambda$ thus we have an inclusion of semigroups $\Lambda \to \bar\Lambda := \mathbf{R}_{\geq 0}\Lambda \cap (\Lambda+(-\Lambda))$, defining an associated toric modification $Z^{\bar\Lambda} \to Z^\Lambda$, which is the *normalization map*. The cone $\mathbf{R}_{\geq 0}\Lambda$ has a vertex if and only if there exists a zero dimensional orbit, and in this case this orbit is reduced to the point of $0 \in Z^\Lambda$ defined by the maximal ideal $\mathfrak{m}_\Lambda := (\Lambda - \{0\})\mathbf{C}[\Lambda]$. The ring $\mathbf{C}[[\Lambda]]$ is the completion of the local ring of germs of holomorphic functions at $(Z^\Lambda,0)$ with respect to its maximal ideal. (See [9] for non necessarily normal toric varieties). In particular, if σ is a cone in the fan Σ, the semigroup $\sigma^\vee \cap M$ is of finite type, it spans the lattice M and the variety $Z^{\sigma^\vee\cap M}$, which we denote also by $Z_{\sigma,N}$ or by Z_σ when the lattice is clear from the context, is normal. If $\sigma \subset \sigma'$ are cones in the fan Σ then we have an open immersion $Z_\sigma \subset Z_{\sigma'}$; the affine varieties Z_σ corresponding to cones in a fan Σ glue up to define the *toric variety* Z_Σ. The torus $(\mathbf{C}^*)^{\operatorname{rk} N}$ is the open dense subset $Z_{\{0\}} \subset Z_\sigma$, and acts on Z_σ for each $\sigma\in\Sigma$; these actions paste into an action on Z_Σ. We say that a fan Σ' is a *subdivision* of the fan Σ if both fans have the same support and if every cone of Σ' is contained in a cone of Σ. The subdivision Σ' defines the *toric modification* $\pi_{\Sigma'} : Z_{\Sigma'} \to Z_\Sigma$ which is equivariant and induces an isomorphism between the tori.

We introduce, for each $\sigma\in\Sigma$, the closed subset $\mathbf{O}_\sigma$ of Z_σ defined by the ideal $(X^w / w \in (\sigma^\vee - \sigma^\perp)\cap M)$ of $\mathbf{C}[\sigma^\vee\cap M]$. The coordinate ring of $\mathbf{O}_\sigma$ is $\mathbf{C}[\sigma^\perp \cap M]$. The map that applies a cone σ in the fan Σ to the set $\mathbf{O}_\sigma \subset Z_\Sigma$ is a bijection between the relative interiors of the cones of the fan and the

orbits of the torus action which inverses inclusions of the closures. The set $\mathbf{O}_\sigma$ is the orbit of the *special point* o_σ defined by $X^u(o_\sigma) = 1$ for all $u \in \sigma^\perp \cap M$. We have that $\dim \mathbf{O}_\sigma = \operatorname{codim} \sigma$, in particular if τ is an edge of Σ the closure $D(\tau)$ of the orbit $\mathbf{O}_\tau$ in Z_Σ is a divisor.

If $\dim \sigma = \operatorname{rk} N$, the orbit $\mathbf{O}_\sigma$ is reduced to the special point o_σ of the affine toric variety Z_σ. The toric variety defined by the fan formed by the faces of the cone σ coincides with the affine toric variety Z_σ. If Σ is a subdivision of σ we have that the exceptional fiber of the modification $\pi_\Sigma : Z_\Sigma \to Z_\sigma$ is equal to $\pi_\Sigma^{-1}(o_\sigma) = \bigcup_{\tau \in \Sigma, \overset{\circ}{\tau} \subset \overset{\circ}{\sigma}} \mathbf{O}_\tau$ (see Proposition page 199, [14]). Any non empty set $I \subset \sigma^\vee \cap M$ defines an integral polyhedron in $M_{\mathbf{R}}$ as the convex hull of the set $\bigcup_{a \in I} a + \sigma^\vee$. We denote this polyhedron by $\mathcal{N}_\sigma(I)$ or by $\mathcal{N}(I)$ if the cone σ is clear from the context. The face of $\mathcal{N}(I)$ determined by $\eta \in \sigma$ is the set $\mathcal{F}_\eta := \{v \in \mathcal{N}(I) / \langle \eta, v \rangle = \inf_{v' \in \mathcal{N}(I)} \langle \eta, v' \rangle\}$. All faces of $\mathcal{N}(I)$ are of this form, the compact faces are defined by vectors in $\overset{\circ}{\sigma}$. The *dual Newton diagram* $\Sigma(I)$ associated to $\mathcal{N}(I)$ is the subdivision of σ formed by the cones $\sigma(\mathcal{F}) := \{\eta \in \sigma \ / \langle \eta, v \rangle = \inf_{v' \in \mathcal{N}(I)} \langle \eta, v' \rangle, \ \forall v \in \mathcal{F}\}$, for $\mathcal{F}$ running through the faces of $\mathcal{N}(I)$. If $\Sigma = \Sigma(I)$, the modification $\pi_\Sigma : Z_\Sigma \to Z_\sigma$ is the *normalized blowing up* of Z_σ centered at the monomial ideal defined by I in $\mathbf{C}[\sigma^\vee \cap M]$ (see [17], Chapter I, section 2). The support of a series $\phi = \sum c_u X^u$ in $\mathbf{C}[[\sigma^\vee \cap M]]$ is the set of exponents of monomials with non zero coefficient. The *Newton polyhedron* $\mathcal{N}(\phi)$ is the integral polyhedron associated to the support of ϕ. If $\eta \in \sigma$ we denote by $\phi_{|\eta}$ the *symbolic restriction* $\sum_{u \in \mathcal{F} \cap M} c_u X^u$ of ϕ to the face $\mathcal{F}$ determined by η.

2 Quasi-ordinary hypersurface singularities

A germ of algebroid hypersurface $(S, 0) \subset (\mathbf{C}^{d+1}, 0)$ is *quasi-ordinary* if there exists a finite morphism $(S, 0) \to (\mathbf{C}^d, 0)$ (called a *quasi-ordinary projection*) such that the discriminant locus is contained (germ-wise) in a normal crossing divisor. In suitable coordinates depending on this projection, the hypersurface $(S, 0)$ has an equation $f = 0$, where $f \in \mathbf{C}[[X]][Y]$ is a *quasi-ordinary polynomial*: a Weierstrass polynomial with discriminant $\Delta_Y f$ of the form $\Delta_Y f = X^\delta \epsilon$, where ϵ is a unit in the ring $\mathbf{C}[[X]]$ of formal power series in the variables $X = (X_1, \dots, X_d)$ and $\delta \in \mathbf{Z}^d_{\geq 0}$. We will suppose from now on that the germ $(S, 0)$ is analytically irreducible at the origin, i.e., the polynomial f is irreducible. The Jung-Abhyankar theorem guarantees that the roots of f, the associated quasi-ordinary branches, are fractional power series in the ring $\mathbf{C}[[X^{1/m}]]$ for some $m \in \mathbf{Z}_{\geq 0}$, (see [1]). If the series $\{\zeta^{(l)}\}_{l=1}^{\deg f} \subset \mathbf{C}[[X^{1/m}]]$ are the roots of f, the discriminant of f is equal to $\Delta_Y f = \prod_{i \neq j} (\zeta^{(i)} - \zeta^{(j)})$ hence each factor $\zeta^{(t)} - \zeta^{(r)}$ is of the form a monomial times a unit in $\mathbf{C}[[X^{1/m}]]$. These monomials (resp. their exponents) are called *characteristic* or *distinguished*. The characteristic exponents $\lambda_1, \dots, \lambda_g$ of f can be labeled in such a way that $\lambda_1 \leq \cdots \leq \lambda_g$ coordinate-wise (see [19]). Since the polynomial f is irreducible, we can identify the analytic algebra of the germ $(S, 0)$ with the ring $\mathbf{C}[[X]][\zeta]$, for ζ any fixed quasi-ordinary branch ζ parametrizing $(S, 0)$.

We denote the lattice $\mathbf{Z}^d$ by M and by M_j the lattice $\mathbf{Z}^d + \sum_{\lambda_i < \lambda_{j+1}} \mathbf{Z} \lambda_i$, for $j = 1, \dots, g$ with the convention $\lambda_{g+1} = +\infty$; the index n_j of the lattice M_{j-1} in M_j non zero, for $j = 1, \dots, g$ (see [12], [13] and [19]). We denote by N (resp. by N_j) the dual lattice of M (resp. of M_j, for $j = 1, \dots, g$) and by $\rho \subset N_{\mathbf{R}}$ the cone spanned by the dual basis of the canonical basis of M. With these notations we have $\mathbf{C}[[X]] = \mathbf{C}[[\rho^\vee \cap M]]$. Since the exponents of the quasi-ordinary branch ζ belong to the

semigroup $\rho^\vee \cap M_g$ we obtain a ring extension $\mathbf{C}[[\rho^\vee \cap M]][\zeta] \to \mathbf{C}[[\rho^\vee \cap M_g]]$ which is the inclusion in the integral closure (see Proposition 14, [13]). Geometrically, the germ of toric variety (Z_{ρ,N_g}, o_ρ) is the normalization $(\bar{S}, 0)$ of $(S, 0)$; more generally the normalization of a germ of quasi-ordinary singularity, non necessarily hypersurface, is a toric singularity (see [25]). In our case, the monomial ideal $\mathfrak{m}_{\rho^\vee \cap M_g} := (\rho^\vee \cap M_g - \{0\})\mathbf{C}[[\rho^\vee \cap M_g]]$ is the maximal ideal of the closed point $0 \in \bar{S}$.

3 The invariance of the semigroups

Let $(X, 0)$ be an analytically irreducible algebroid germ. Denote by $p : (\bar{X}, 0) \to (X, 0)$ the normalization map and by $b : (B, E) \to (\bar{X}, 0)$ the normalized blow up centered at $0 \in \bar{X}$. If D is any irreducible component of the exceptional divisor E of the modification b we denote by ν_D the associated divisorial valuation of the field of fractions K of the analytic algebra R of the germ $(X, 0)$. We have that $\nu_D(h)$ is equal to the vanishing order of $h \circ b$ at the component D, for $h \in K - \{0\}$. The valuation ν_D defines a filtration of R with ideals $\mathfrak{p}_k = \{\phi \in R / \nu_D(\phi) \geq k\}$ for $k \geq 0$. We denote by $\mathrm{gr}_D(X, 0)$ the associated graded ring $\mathrm{gr}_D(X, 0) := \bigoplus_{k \in \mathbf{Z}_{\geq 0}} \mathfrak{p}_k / \mathfrak{p}_{k+1}$. The graded ideal $\mathfrak{m}_D(X, 0) := \bigoplus_{k \in \mathbf{Z}_{\geq 1}} \mathfrak{p}_k / \mathfrak{p}_{k+1}$ is maximal and the pair $(\mathrm{gr}_D(X, 0), \mathfrak{m}_D(X, 0))$ depends only on the analytic algebra R and the exceptional divisor D.

Suppose that $(\bar{X}, 0)$ is the germ of an affine toric variety $Z_{\rho,N}$ at its zero dimensional orbit (which is assumed to exists). The normalized blow up of the germ $(\bar{X}, 0) = (Z_{\rho,N}, o_\rho)$ centered at $0 = o_\rho$ is the toric modification $\pi_\Sigma : Z_\Sigma \to Z_{\rho,N}$ where $\Sigma = \Sigma(\rho^\vee \cap M - \{0\})$. By definition, the modification π_Σ is an isomorphism outside the origin $o_\rho \in Z_{\rho,N}$ hence the exceptional divisor E is equal to the exceptional fiber $\pi_\Sigma^{-1}(o_\rho)$. The irreducible components of this divisor are of the form $D(\tau)$, where τ runs through the edges of Σ with $\overset{\circ}{\tau} \subset \overset{\circ}{\rho}$. There exists at least one such edge τ, we fix it and we denote $D(\tau)$ by D and by n the integral lattice vector of N in the edge τ. The following property of the divisorial valuation associated to a toric divisor is used by Gonzalez-Sprinberg and Bouvier in the algebraic case (see [3] and [2]).

Lemma 3.1 *If $0 \neq \phi = \sum c_u X^u \in \mathbf{C}[[\rho^\vee \cap M]]$ then $\nu_D(\phi) = \min_{c_u \neq 0} \langle n, u \rangle$.*

Proof. Let v denote any vertex of the compact face $\mathcal{F}$ defined by n on the polyhedron $\mathcal{N}(\phi)$, we have that ϕ factors by X^v in a neighborhood of the divisor D in the open chart Z_τ of Z_Σ. We have that $X^{-v}\phi = X^{-v}\phi_{|\mathcal{F}} + r$ and the exponents of the terms in r belong to $\tau^\vee - \tau^\perp$, thus r vanish on $\mathbf{O}_\tau$ and then on D. We have also that $\nu_D(X^{-v}\phi_{|\mathcal{F}}) = 0$ since $X^{-v}\phi_{|\mathcal{F}}$ is a polynomial in $\mathbf{C}[\tau^\vee \cap M]$ with non zero constant term. We deduce from these facts that $\nu_D(\phi) = \nu_D(\phi_{|\mathcal{F}}) = \nu_D(X^v)$ which is equal to $\langle n, v \rangle = \min_{c_u \neq 0} \langle n, u \rangle$ (see [7], page 61). $\square$

3.1 The toric case

Suppose that $(X, 0)$ is the germ of a non necessarily normal affine toric variety Z^Λ at its zero orbit 0 (which is assumed to exist). The normalization $Z^{\bar{\Lambda}}$ is the affine toric variety $Z^{\bar{\Lambda}} = Z_{\rho,N}$, where N is the dual lattice of $\Lambda + (-\Lambda)$ and $\rho \subset N_{\mathbf{R}}$ is the dual cone of $\mathbf{R}_{\geq 0}\Lambda$. We consider the graduation $\mathbf{C}[\Lambda]^{(n)}$

of the algebra $\mathbf{C}[\Lambda]$, with homogeneous terms $H_k := \bigoplus_{u\in\Lambda,\langle n,u\rangle=k} \mathbf{C}X^u$, for $k \in \mathbf{Z}_{\geq 0}$ induced by the primitive vector $n \in \overset{\circ}{\tau}$ associated to the exceptional divisor D. We denote by $\mathfrak{m}_\Lambda^{(n)}$ the maximal graded ideal $\mathfrak{m}_\Lambda^{(n)} := \bigoplus_{k\geq 1} H_k$ and by $\mathfrak{m}_\Lambda = (\Lambda - \{0\})\mathbf{C}[\Lambda]$ the same ideal without the graded structure.

Proposition 3.2 *The pair* $(\mathrm{gr}_D(Z^\Lambda, 0), \mathfrak{m}_D(Z^\Lambda, 0))$ *is isomorphic to* $(\mathbf{C}[\Lambda]^{(n)}, \mathfrak{m}_\Lambda^{(n)}))$.

Proof. The analytic algebra of the algebroid germ $(X, 0)$ is isomorphic to $\mathbf{C}$-algebra $\mathbf{C}[[\Lambda]]$. It is sufficient to prove that the map $\mathfrak{p}_k/\mathfrak{p}_{k+1} \to H_k$ given by $\phi \in \mathbf{C}[[\Lambda]]$, $\phi \in \mathfrak{p}_k$,

$$0 \neq \phi \bmod \mathfrak{p}_{k+1} \mapsto \phi_{|n}, \text{ and } 0 \bmod \mathfrak{p}_{k+1} \mapsto 0, \tag{1}$$

is well defined and extends to a graded isomorphism $\mathrm{gr}_D(Z^\Lambda, 0) \to \mathbf{C}[\Lambda]^{(n)}$.

We have that if $0 \neq \phi \in \mathbf{C}[[\Lambda]]$ and if $\eta \in \overset{\circ}{\rho}$ the symbolic restriction $\phi_{|\eta}$ belongs to $\mathbf{C}[\Lambda]$, conversely given any $u \in \Lambda$ there exists $\phi = X^u \in \mathbf{C}[[\Lambda]]$ such that $\phi_{|\eta} = X^u$. It follows that $H_k - \{0\}$ is equal to $\{\phi_{|n}/\phi \in \mathfrak{p}_k|\mathfrak{p}_{k+1}\}$ (where $\mathfrak{p}_k|\mathfrak{p}_{k+1} = \{\phi \in \mathfrak{p}_k/\phi \notin \mathfrak{p}_{k+1}\}$) . If $0 \neq \phi, \phi' \in \mathfrak{p}_k|\mathfrak{p}_{k+1}$ we have that $\phi_{|n} = {\phi'}_{|n}$ if and only if $\phi = \phi' \bmod \mathfrak{p}_{k+1}$ since the terms which may differ on ϕ and ϕ', viewed in $\mathbf{C}[[\Lambda]]$, have exponents of the form, u with $\langle n, u\rangle > k$. We obtain that the map $\mathfrak{p}_k/\mathfrak{p}_{k+1} \to H_k$ is an isomorphism of vector spaces over $\mathbf{C} = H_0 = \mathfrak{p}_0/\mathfrak{p}_1$. The face of the Newton polyhedron $\mathcal{N}(\phi\psi)$ defined by n is equal to the Minkowski sum of the faces defined by n on the polyhedra $\mathcal{N}(\phi)$ and $\mathcal{N}(\psi)$ (see [6], page 105). We deduce from this that $(\phi\psi)_{|n} = \phi_{|n}\psi_{|n}$ and therefore that the map $\mathrm{gr}_D(Z^\Lambda, 0) \to \mathbf{C}[\Lambda]^{(n)}$ is a graded isomorphism. □

We deduce from this result a local version of the *isomorphism problem* for germs of toric varieties (see [15] for the affine case). A topological proof of the following corollary has been given by Popescu-Pampu in [25] in the case of normal simplicial toric singularities.

Corollary 3.3 *Let* Z^Λ *and* $Z^{\Lambda'}$ *be two non necessarily normal affine toric varieties with zero orbits* 0 *and* $0'$ *respectively. If there exists an isomorphism of algebroid germs* $(Z^\Lambda, 0) \cong (Z^{\Lambda'}, 0')$ *the semigroups* Λ *and* Λ' *are isomorphic.*

Proof. For any component D of the exceptional divisor E the pair obtained from $(\mathrm{gr}_D(Z^\Lambda, 0), \mathfrak{m}_D(Z^\Lambda, 0))$ by forgetting the graduation is isomorphic to $(\mathbf{C}[\Lambda], \mathfrak{m}_\Lambda)$, thus it does not depend on the divisor D. We obtain isomorphic pairs of $\mathbf{C}$-algebra and maximal ideal $(\mathbf{C}[\Lambda], \mathfrak{m}_\Lambda)$ and $(\mathbf{C}[\Lambda'], \mathfrak{m}_{\Lambda'})$. We apply Gubeladze's theorem 2.1 case (a) [15], to deduce the existence of an isomorphism of semigroups $\Lambda \cong \Lambda'$. □

3.2 The quasi-ordinary hypersurface case

We suppose now that $(X, 0)$ is the germ of quasi-ordinary hypersurface $(S, 0)$. In [12] the semigroup $\Gamma := \mathbf{Z}^d_{\geq 0} + \gamma_1\mathbf{Z}_{\geq 0} + \cdots + \gamma_g\mathbf{Z}_{\geq 0} \subset \rho^\vee \cap M_g$ where $\gamma_1 = \lambda_1$ and $\gamma_{j+1} = n_j\gamma_j + \lambda_{j+1} - \lambda_j$ for $j = 1, \ldots, g-1$, is associated to a fixed quasi-ordinary branch ζ parametrizing the quasi-ordinary hypersurface germ $(S, 0)$ (see [11]) following the analogy to the case of plane curves (see [28]). The zero dimensional orbit 0 of

the affine toric variety Z^Γ corresponds to the maximal ideal $\mathfrak{m}_\Gamma := (\Gamma - \{0\})\mathbf{C}[\Gamma]$. In [12] it is shown that the normalizations of $(Z^\Gamma, 0)$ and $(S,0)$ coincide. We have that $(\bar{X}, 0) = (Z^{\bar{\Gamma}}, 0) = (Z_{\rho, N_g}, o_\rho)$, with the notations of section 1 and 2.

Proposition 3.4 *For any irreducible component D of the exceptional divisor of the normalized blow up of $(\bar{X}, 0)$ centered at 0, the pairs of graded ring and maximal graded ideal $(\mathrm{gr}_D(S,0), \mathfrak{m}_D(S,0))$ and $(\mathrm{gr}_D(Z^\Gamma, 0), \mathfrak{m}_D(Z^\Gamma, 0))$ are isomorphic.*

Proof. An irreducible component D of the exceptional divisor of the normalized blow up of $(\bar{X}, 0)$ centered at 0, corresponds to a primitive integral vector $n \in N_g$ in the interior of the cone ρ. The analytic algebra of $(S,0)$ is isomorphic to $\mathbf{C}[[\rho^\vee \cap M]][\zeta]$, for ζ a fixed quasi-ordinary branch parametrizing $(S,0)$. We consider the graded ring $\mathbf{C}[\Gamma]^{(n)}$. By proposition 3.2 it is sufficient to prove that the map $\mathfrak{p}_k/\mathfrak{p}_{k+1} \to H_k$ given by formula (1) for $\phi \in \mathbf{C}[[\rho^\vee \cap M]][\zeta]$, $\phi \in \mathfrak{p}_k$, is well defined and extends to a graded isomorphism $\mathrm{gr}_D(S,0) \to \mathbf{C}[\Gamma]^{(n)}$.

We have that if $0 \neq \phi \in \mathbf{C}[[\rho^\vee \cap M]][\zeta]$ and if $\eta \in \overset{\circ}{\rho}$ the symbolic restriction $\phi_{|\eta}$ belongs to $\mathbf{C}[\Gamma]$, conversely given any $u \in \Gamma$ there exists $\phi \in \mathbf{C}[[\rho^\vee \cap M]][\zeta]$ such that $\phi_{|\eta} = X^u$ (see Proposition 2.16 [12] or Proposition 3.1 and Theorem 3.6 of [11]). This property allows us to proceed with proof of the statement exactly in the same way as in proposition 3.2. $\Box$

Corollary 3.5 *If ζ and ζ' are two quasi-ordinary branches parametrizing the germ $(S,0)$ then there exists an isomorphism of the corresponding semigroups Γ and Γ'.*

Proof. For any irreducible component D of the exceptional divisor of the normalized blow-up of $(\bar{S}, 0)$, the pair of $\mathbf{C}$-algebra and maximal ideal obtained from $(\mathrm{gr}_D(S,0), \mathfrak{m}_D(S,0))$ by forgetting the graduation does not depend on the component D. Indeed, it is isomorphic to $(\mathbf{C}[\Gamma], \mathfrak{m}_\Gamma)$ and to $(\mathbf{C}[\Gamma'], \mathfrak{m}_{\Gamma'})$ by propositions 3.2 and 3.4. We apply Gubeladze's theorem 2.1 case (a) [15], to deduce the existence of an isomorphism of semigroups $\Gamma \cong \Gamma'$. $\Box$

Remark 1 Proposition 3.4 and corollary 3.5 extend to the case of toric quasi-ordinary hypersurfaces in [13]. The arguments used in the proof of proposition 3.4 above correspond to Proposition 31 Section 4.1 and Proposition 34 Section 4.2 in [13].

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